

Soliton Solutions of the Integrable Chiral Model in 2+1 Dimensions ¹

THEODORA IOANNIDOU

*Dept of Mathematical Sciences, University of Durham,
Durham DH1 3LE, UK*

Abstract We present soliton and soliton-antisoliton solutions for the integrable chiral model in 2+1 dimensions with nontrivial (elastic) scattering. These solutions can be obtained either as the limiting cases of the ones already constructed by Ward or by adapting Uhlenbeck's method.

1 Introduction

The integrable chiral model was derived by Ward [1], by dimensional reduction from the self-dual-Yang-Mills equations in (2+2) dimensions, and is defined as:

$$(\eta^{\mu\nu} + \varepsilon^{\mu\nu\alpha} V_\alpha) \partial_\mu (J^{-1} \partial_\nu J) = 0, \quad (1)$$

where J is a map of \mathbf{R}^{2+1} into $SU(2)$, $\eta^{\mu\nu} = \text{diag}(-1, 1, 1)$, and V_α is a unit vector in space-time. In Ward's model V_α is a space-like vector, i.e. $V_\alpha = (0, 1, 0)$. [This choice was motivated by the energy conservation.] Notice that the standard chiral model in (2+1) dimensions has $V_\alpha = (0, 0, 0)$.

Ward showed that the integrable chiral model has the same conserved energy density as the standard chiral model. This energy is given by

$$\mathcal{E} = -\frac{1}{2} \text{tr} \left[(J^{-1} J_t)^2 + (J^{-1} J_x)^2 + (J^{-1} J_y)^2 \right]. \quad (2)$$

In order to ensure the finiteness of the energy of the solutions, we require that $J \rightarrow K + O(r^{-1})$ at spatial infinity, with $x + iy = r e^{i\theta}$.

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First of all, recall that the standard chiral model is not integrable but Lorentz covariant. By contrast, the existence of V_α in (1) breaks explicitly the Lorentz covariance of the model but it makes it integrable; it is associated with a linear system (see (5)), passes the Painleve test, has an inverse scattering description and admits multisoliton solutions.

One might expect the scattering of solitons of (1) to be trivial, due to its integrability. However, numerical simulation show that right-angle scattering (like the ones observed to nonintegrable models) can occur. This happens since the solitons in this model possess internal degrees of freedom that determines their orientation in space. So they can interact trivially and nontrivially depending on the orientation of these internal parameters and on the value of the impact parameter. And due to the integrability of the model these solutions should be constructed explicitly. From now on, by solitons we shall mean localized energy configurations that move but we shall not imply stability of the shape or the velocity or simple behaviour in collision. In fact, the solutions we are going to discuss here, are localized along the direction of motion; they are not, however, of constant size; their height, which is the maximum of the energy density \mathcal{E} , is time dependent.

2 Construction of Soliton Solutions

Harmonic maps from \mathbf{R}^2 into Lie groups were studied by Uhlenbeck [2], who in a seminal work, showed that all solutions of the integrable *static* standard chiral model can be factorized into a product of factors (so-called *n-unitons*), of the form

$$J = \prod_{i=1}^n (1 - 2R_i), \quad (3)$$

where $R_i = (q_i^\dagger \otimes q_i) / |q_i|^2$ are Hermitian projectors, and q_i are 2-dimensional vectors. Using Lax pair she show that the projectors R_i have to satisfy first-order partial differential equations. This can be extended to the nonstatic case (1).

So the 2-uniton solution of (1) has the form of (3) with

$$q_1 = (1, f), \quad q_2 = (1 + |f|^2)(1, f) - 2i(tf' + h)(\bar{f}, -1), \quad (4)$$

where f and h are rational meromorphic functions of $z = x + iy$. For $f = z^p$ and $h = z^q$ (for $p > q$), the configuration consists of $(p - 1)$ static solitons at the centre of mass of the

system, accompanied by $N = q - p + 1$ solitons accelerating towards the ones in the middle, scattering at an angle of π/N , and then decelerating as they separate. This follows from the fact that the field J departs from its asymptotic value J_0 when $(tf' + h) \rightarrow 0$ which is true when either $z^{(p-1)} = 0$ or $tp + z^N = 0$; and this is approximately where the solitons are located.

Let us indicate how these solutions were constructed, and how others may be obtained. Equation (1) is the consistency condition for the Lax pair

$$L\Psi \equiv (\lambda\partial_x - \partial_y - \partial_t)\Psi = A\Psi, \quad M\Psi \equiv (\lambda\partial_t - \lambda\partial_y - \partial_x)\Psi = B\Psi, \quad (5)$$

where A and B are 2×2 matrices independent of λ , and the integrability condition implies that $\Psi(\lambda, x, y, t)|_{\lambda=0} = J^{-1}(t, x, y)$.

- Uhlenbeck's construction, assumes that

$$\Psi = \mathcal{K} \left(1 - \frac{2i}{\lambda - i} R_n\right) \dots \left(1 - \frac{2i}{\lambda - i} R_2\right) \left(1 - \frac{2i}{\lambda - i} R_1\right), \quad (6)$$

while the restriction for $A = (L\Psi)\Psi^{-1}$ and $B = (M\Psi)\Psi^{-1}$ to be independent of λ : impose a sequence of first-order differential equations for time dependent projector valued fields.

So, to construct time-dependent solutions of the $SU(N)$ model (1), one can start from a constant solution (0-uniton) and add to it 1-uniton. This solution will be nonconstant - but it is static. Then, to this uniton, one can add a second uniton, and so on. In [3] we give a 2-uniton solution of the $SU(3)$ model (1). In fact, the number of unitons can be arbitrary. In the static $SU(2)$ case considered by Uhlenbeck the only solutions are those described by constant matrices (0-uniton) and factors constructed from holomorphic functions (1-uniton). In the model (1) the 0- and 1-uniton solutions are the same but then, we have further ones corresponding to 2- and more unitons (cf. [4]). These additional solutions are nonstatic. We do not know at this stage, whether there is any bound on the uniton number so that all solutions correspond to field configurations of up this number.

- Ward [1], on the other hand, using the *Riemann problem with zeroes*, proved that Ψ is of the form

$$\Psi = 1 - \sum_{k=1}^n (\lambda - \mu_k)^{-1} \left(\sum_{l=1}^n (\Gamma^{-1})^{kl} \bar{m}_\alpha^l m_b^k \right), \quad \Gamma^{kl} = (\bar{\mu}_k - \mu_l)^{-1} \sum_{\alpha=1}^2 \bar{m}_\alpha^k m_b^l \quad (7)$$

where $m_\alpha^k = (1, f_k)$ and f_k is a rational function of a complex parameter ω_k ($\omega_k|_{\mu_k=i} \rightarrow z$ which correspond to a static J). These solutions pass each other without any change of

direction or phase shift. All this assumes that the parameters μ_k are distinct and $\mu_k \neq \bar{\mu}_l$. The solution (6) is derived from (7) for $n = 2$, by setting $\mu_1 = i + \varepsilon$, $\mu_2 = i - \varepsilon$, $f_1 = f + \varepsilon h$, $f_2 = f - \varepsilon h$ and by taking the limit $\varepsilon \rightarrow 0$ (cf. [5], [4]). It may seem strange that one can take the limit of a family of soliton solutions with trivial scattering, and obtain a new one with nontrivial scattering. However, in [4] it has been shown that as $\varepsilon \rightarrow 0$ the solitons disperse, shift and interact with each other.

A different and maybe more interesting problem is the existence of soliton-antisoliton configurations for (1). Roughly speaking, solitons correspond to f being a function of z and antisolitons correspond to a function of \bar{z} . One way of proceeding is to take (7) with $n = 2$, put $\mu_1 = i + \varepsilon$ and $\mu_2 = -i - \varepsilon$ and take the limit $\varepsilon \rightarrow 0$ (cf. [4]). In order Ψ to be smooth on \mathbf{R}^{2+1} , it is necessary to take $f_1 = f$ and $f_2 = \bar{f}^{-1} - \varepsilon h$ with $f = f(z)$ and $h = h(\bar{z})$. The corresponding field J describes solitons and antisolitons that are well separated, accelerate towards each other until they merge at the origin and scatter at right angles as they separate again (no radiation emission). This is the first example in (2+1) dimensions with elastic scattering between soliton-antisoliton solution. [Recall that for the pure O(3) σ -model the soliton and antisoliton after the collision annihilate into a wave of pure radiation.]

A topological charge may be defined for the field J by exploiting the connection of (1) with the O(3) σ -model (recall that (1) is not a topological model). The standard chiral model is equivalent to the O(4) σ -model, through the relation

$$J = 1 \phi_0 + i \boldsymbol{\sigma} \cdot \boldsymbol{\phi}. \quad (8)$$

For the O(4) model, the field at fixed time is a map $(\phi_0, \boldsymbol{\phi}) : \mathbf{S}^3 \rightarrow \mathbf{S}^3$, i.e. $\pi_2(\mathbf{S}^3) = 0$ so there is no winding number. However, for soliton solutions that correspond to some initial embedding of O(3) space into O(4), there is a useful topological quantity, as we are going to see. For the O(3) model, the field is a map $\phi : \mathbf{S}^2 \rightarrow \mathbf{S}^2$ and due to the homotopy relation $\pi_2(\mathbf{S}^2) = \mathbb{Z}$, such maps are classified by an integer winding number, given by

$$\mathcal{N} = (8\pi)^{-1} \int \epsilon_{ij} \boldsymbol{\phi} \cdot (\partial_i \boldsymbol{\phi} \wedge \partial_j \boldsymbol{\phi}) d^2x, \quad (9)$$

So, if the field J (at fixed time) is restricted to an \mathbf{S}^2 equator of the \mathbf{S}^3 target space, while it never maps to the antipodal points $\{1, -1\}$, one can define a winding number for (1). An expression for this winding number is easy to be given, since it is the winding number

of the map after projection onto the chosen \mathbf{S}^2 equator, i.e. (9) where $\phi \rightarrow \phi' \equiv \phi/|\phi|$.

So we see that the integrable chiral model has many interesting solutions and these solutions can be derived using *analytic* methods. These structures travel with nonconstant velocities, their size is not constant, and they interact nontrivially (like the nonintegrable models). The model does not possess rotational symmetry in the x, y plane; however, most explicit solutions seem to possess. Such results might be useful for connecting integrable and nonintegrable models, which possess soliton solutions. In addition, they indicate the likely occurrence of new phenomena in higher-dimensional soliton theory that are not present in 1+1 dimensions.

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